PARTIAL DIFFERENTIAL EQUATION

Partial Differential Equation

An equation containing partial differential coefficients is called a Partial Differential Equation . A partial differential equation involves two or more independent variables and one dependent variable.

Order : The order of a partial differential equation is that of the highest order derivative occurring in that equation.

Degree : The degree of a partial differential equation is the degree of the highest order partial derivative occurring in the equation.

Form a p.d.e by eliminating the arbitrary constants a and b from

$$\mathbf{z = (x+a)^2 + (y+b)^2}$$

solution :

Given $z = (x+a)^2 + (y+b)^2$ -------(1)

$$p = \frac{\partial z}{\partial x} = 2(x+a) \; i.e., x + a = \frac{p}{2}$$

$$q = \frac{\partial z}{\partial y} = 2(y-b) \; i.e., y - b = \frac{q}{2}$$

$$\therefore (1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$$4z = p^2 + q^2$$

which is the required p.d.e.

Form the partial differential equation by eliminating a and b from

$$\boldsymbol{z = a(x+y) + b}$$

solution :

$$z = a(x+y) + b$$

$$p = \frac{\partial z}{\partial x} = a \qquad \text{--------(1)}$$

$$q = \frac{\partial z}{\partial y} = a \qquad \text{--------(2)}$$

From (1) & (2) we get the required p.d.e

$$Z = p(x+y) + q$$

Eliminate the arbitrary function f from $z = f\left(\frac{y}{x}\right)$ ***and form a p.d.e***

solution :

Given $z = f\left(\frac{y}{x}\right)$

Diff (1) p.w.r.to ‘x’ we get

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \quad \text{--------- (2)}$$

Diff (1) p.w.r.to ‘y’ we get

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad \text{---------- (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{-y}{x}$$

$$Therefore\ px = qy$$

i.e., px +qy =0 is the required p.d.e.

Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - 4\frac{\partial^3 z}{\partial x \partial y^2} + 8\frac{\partial^3 z}{\partial y^3} = 0$

solution :

Given $(D^3 - 2D^2D' - 4DD'^2 + 8D'^3)z = 0$

The A.E. is $m^3 - 2m^2 - 4m + 8 = 0$

(Replace D by m and D' by 1)

$$m^2(m-2) - 4(m-2) = 0$$

$$(m-2)(m^2-4) = 0$$

$$(m-2) = 0, (m^2-4) = 0$$

$$m = 2, m = \pm 2$$

$$\therefore m = 2, 2, -2$$

$$\therefore z = \varphi_1(y+2x) + \varphi_2(y+2x) + \varphi_3(y-2x)$$

Find the complete integral of $\boldsymbol{z = px + qy - 2\sqrt{pq}}$

solution :

Given $z = px + qy - 2\sqrt{pq}$

This is of the form $z = px + qy + f(p, q)$ (Clairaut,s form)

Hence the complete integral is $z = ax + by - 2\sqrt{ab}$

Where a & b are arbitrary constants.

Write the solution of $px^2 + qy^2 = z^2$

solution :

Given $px^2 + qy^2 = z^2$

This is equation of the form Pp = Qq = R

Where $P = x^2$, $Q = y^2$, $R = z^2$

Lagrange's subsidiary equations are $\frac{dx}{P} + \frac{dy}{Q} + \frac{dz}{R}$

i.e. $\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}$ --------- (1)

Take $\frac{dx}{x^2} = \frac{dy}{y^2}$

Take $\frac{dy}{y^2} = \frac{dz}{z^2}$

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2} = \int \frac{dz}{z^2}$$

$$\frac{-1}{x} = \frac{-1}{y} - c_1 \qquad \frac{-1}{y} = \frac{-1}{z} - c_2$$

$$\frac{1}{y} - \frac{1}{x} = -c_1 \qquad \frac{1}{y} - \frac{1}{z} = c_2$$

$$c_1 = \frac{1}{x} - \frac{1}{y}$$

i.e., $v = \frac{1}{y} - \frac{1}{z}$

i.e., $u = \frac{1}{x} - \frac{1}{y}$

Hence the general solution is $f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$

Find the particular integral of $(D^2 - DD')z = \sin x \cos 2y$

solution :

$$P.I = \frac{1}{(D^2 - DD')} \sin x \cos 2y$$

$$= \frac{1}{(D^2 - DD')} \frac{1}{2}[\sin(x + 2y) + \sin(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{(D^2 - DD')} \sin(x + 2y)\right] + \frac{1}{2}\left[\frac{1}{(D^2 - DD')} \sin(x - 2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1 - (-2)} \sin(x + 2y)\right] + \frac{1}{2}\left[\frac{1}{-1 - 2} \sin(x - 2y)\right]$$

$$= \frac{1}{2}[\sin(x + 2y)] + \frac{1}{2}\left[\frac{-1}{3} \sin(x - 2y)\right]$$

$$= \frac{1}{2}[\sin(x + 2y)] + \frac{-1}{6}[\sin(x - 2y)]$$

Solve $(D^2 - 2DD' + D'^2)z = cos(x - 3y)$

solution :

Given $(D^2 - 2DD' + D'^2)\, z = \cos(x - 3y)$

The auxillary equation is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1,1$$

$$\text{C.F.} = f_1(y + x) + f_2\,(y + x)$$

$$\text{P.I} = \frac{1}{D^2 - 2DD' + D'^2}\cos(x - 3y)$$

$$= \frac{\cos(x-3y)}{-1-2(3)-9}$$

$$= \frac{-1}{16}\cos(x - 3y)$$

The complete solution is $z = f_1(y + x) + f_2(y + x) - \frac{1}{16}\cos(x - 3y)$