Mathematical Logic - Module-I

Connectives

Notation:

* A declarative sentences which do not broken down further, are called the primary statements.

* A primary statement admits, in the object language, ~~which have~~ one and only one of two possible values, called "truth values."

* The two possible truth values are true and false, denoted by 'T' & 'F' respectively.

* Since it admits only two possible truth values, our logic is also called a two-valued logic.

* Declarative statements in the object language are of two types.

(i) primary statements: It will denoted by capital letters A, B, C, ...

(ii) Second types of sentences are obtained by joining two or more primary statements, using certain symbols (called connectives) and certain punctuation marks, (called parentheses)

* A declarative sentences having a 'unique' possible truth values, are called 'statements'.

* Primary statements are also called as atomic statements or primary statements.

Examples:

1) Canada is a country.
2) $1 + 101 = 110$.

$\Big\}\rightarrow$ statements.

3) This statement is false → NOT a statement. (∵ we cannot assign T or F)
4) Close the door → NOT a statement. (∵ It is a command)

Connectives:

We have three basic connectives, called: (1) Negation; (2) Conjunction; (3) Disjunction.

1. Negation:

⊛ The "negation" of a statement is formed by introducing the word "not" at a proper phase in the given statement.

⊛ If 'P' denotes a statement, then the negation of P is written as '$\neg P$'. (read as 'not P')

⊛ The truth table of '$\neg P$' is given below:

P	$\neg P$
T	F
F	T

⊛ The negation of P ($\neg P$) is also denoted by '$\sim P$', '$\overline{P}$' or 'NOT P'.

Example:

Consider the statement:

P: London is a city.

Then $\neg P$: It is not the case that London is a city.

(or) $\neg P$: London is not a city.

Conjunction:

⊛ 'Conjunction' of two statements is a statement joined by the word 'and'.

⊛ The conjuction of two statements P & Q is denoted by '$P \wedge Q$' (read as "P and Q")

⊛ The statement '$P \wedge Q$' has the truth value 'T' whenever both P and Q have the truth value T; otherwise it has 'F'.

⊛ The truth value of $P \wedge Q$ is given below:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: Consider P: Mark is rich

Q: Mark is happy.

Then, $P \wedge Q$: Mark is rich and he is happy.

(or) $P \wedge Q$: Mark is both rich and happy.

Note:
- Consider, P: "Jack and Jill are cousins."
  Here, the word 'and' is not a conjunction.

3. Disjunction:

- The disjunction of two statements P and Q is the statement $P \vee Q$ (read as "P or Q")
- The statement $P \vee Q$ has the truth value F only when both P & Q have F; otherwise it is 'T'.
- The truth table for disjunction is given below:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

- The word "or" here, means the "inclusive OR".

Remark: The statements which contain one or more primary statements and connectives, are called molecular or composite or compound statements.

For example, $\neg P$, $P \vee Q$, $P \wedge (\neg Q)$ are compound statements.

Conditional statements:

⊛ If P and Q are any two statements, then the statement '$P \to Q$' (read as "If P, then Q") is called a conditional statement.

⊛ P is called the antecedent in $P \to Q$ and Q is called the consequent in $P \to Q$.

⊛ The statement $P \to Q$ has a truth value 'F' only when Q has F and P has T, otherwise it has T.

⊛ The truth value for conditional is as follows:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

⊛ Som representations of '$P \to Q$':

(1) Q is necessary for P.
(2) P is sufficient for Q.
(3) Q if P
(4) P only if Q
(5) P implies Q.

Example: Write the following statement in symbolic form:

"If either Jerry takes Calculus or Ken takes Sociology, then Larry will take English."

Solution:

Denote J : Jerry takes Calculus

K : Ken takes Sociology.

L : Larry takes English.

Then, the given statement can ~~by~~ be symbolized as

"$(J \vee K) \rightarrow L$."

Example 4:

Construct the truth table for $(P \rightarrow Q) \wedge (Q \rightarrow P)$.

Solution:

The truth table for $(P \rightarrow Q) \wedge (Q \rightarrow P)$:

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Biconditional Statement:

⊛ If P and Q are any two statements, then the statement $P \rightleftarrows Q$ (read as "P if and only if Q", & abbriviated as "P iff Q") is called a biconditional statement.

⊛ $P \rightleftarrows Q$ has the truth value 'T' only when both P and Q have same truth values; otherwise, it is 'F'.

⊛ The statement $P \rightleftarrows Q$ is translated as "P is necessary and sufficient for Q".

⊛ The truth table for $P \rightleftarrows Q$ is given by,

P	Q	$P \rightleftarrows Q$
T	T	T
T	F	F
F	T	F
F	F	T

Result:

We can see that the statement $P \rightleftarrows Q$ is identically equivalent to the statement $(P \to Q) \wedge (Q \to P)$

Well-formed formulas: (w.f.f)

A statement formula is an expression which is a string consisting of variables (capital letters), parentheses, and connective symbols. We can give a recursive definition of a statement formula, often called a well-formed formula (w.f.f.)

Rules for wff:

(1) A statement variable standing alone is a wff.

(2) If A is a wff, then $\neg A$ is a w.f.f.

(3) If A and B are well-formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ & $(A \rightleftarrows B)$ are w.f.f's.

(4) A string of symbols containing the statement variables, connectives, and parenthesis is a wff, iff it can be obtained by finitely many applications of the rules (1), (2) & (3).

The following are wff:

$\neg(A \vee B)$, $(P \rightarrow (P \vee Q))$, $(P \rightleftarrows (Q \rightarrow R))$,

$(((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightleftarrows (P \rightarrow R))$.

The following are Not wff:

(1) $\neg P \wedge Q$ [A wff would be either $(\neg P \wedge Q)$ or $\neg(P \wedge Q)$]

(2) $(P \vee Q) \rightarrow (\wedge Q)$ [not a wff, because $\wedge Q$ is not a wff]

(3) $(P \to Q$. [not wff, since one parenthesis in the end is missing. $(P \to Q)$ is a wff]

(4) $P \wedge Q \to Q)$. [not wff; since one parenthesis in the beginning is missing. It should be $((P \wedge Q) \to Q)$]

<u>Note:</u>

For our convenience, we shall omit the outer parenthesis in the wff. For example, we write $(P \wedge Q) \to R$ ~~insted~~ instead of $((P \wedge Q) \to R)$.

<u>Tautology & Contradiction:</u>

[Recall: For our convenience, we use the expression, "the truth value of the statement formula" to mean the entries in the final column of the truth table of the formula.]

A statement formulla which is "true" (T) in all the truth values of the statement, is called a <u>universally valid formula</u> or a <u>tautology</u> or a <u>logical truth</u>.

A statement formula which is "false" (F) in all the truth values of the statement, is called a <u>contradiction</u>.

We say that a statement formula which is a tautology is <u>identically true</u> and a formula which is a contradiction is <u>identically false</u>.

<u>Example:</u>

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

Here $P \vee \neg P$ is a tautology as well as $P \wedge \neg P$ is a contradiction.

Substitution instance:

A formula A is called a <u>substitution instance</u> of another formula B if A can be obtained from B by substituting formulas for some variables of B, with the condition that the same formula is substituted for the same variable each time it occurs.

<u>Illustration:</u>

① Let $B: P \rightarrow (J \wedge P)$. Substitute $R \rightleftarrows S$ for P in B, we get

$$A: (R \rightleftarrows S) \rightarrow (J \wedge (R \rightleftarrows S)).$$

Then A is a substitution instance of B.

Note that $(R \rightleftarrows S) \rightarrow (J \wedge P)$ is not a substitution instance of B. ($\because$ P in $J \wedge P$ is not replaced by $R \rightleftarrows S$)

(2) Some substitution instances of $P \to \neg Q$ are

(2.1) $(R \wedge \neg S) \to \neg(J \vee M)$ [Substitute $R \wedge \neg S$ for P & $J \vee M$ for Q]

(2.2) $(R \wedge \neg S) \to \neg(R \wedge \neg S)$ [Substitute $R \wedge \neg S$ for both P & Q]

(2.3) $(R \vee S) \to \neg Q$ [Substitute $R \vee S$ for P]

(2.4) $Q \to \neg(P \wedge \neg Q)$ [Substitute Q for P and $P \wedge \neg Q$ for Q]

(3) Consider $P \to \neg Q$.

(3.1) $(P \vee Q) \to \neg R$ is a substitution instance of $P \to \neg Q$, by substituting $P \vee Q$ for P & R for Q.

(3.2) $(P \vee R) \to \neg R$ is not a substitution instance for $P \to \neg Q$; but it is a substitution instance of $(P \vee Q) \to \neg Q$, by substituting R for Q.

Important remarks:

* Any substitution instance of a tautology is a tautology.

Note that $P \vee \neg P$ is a tautology; some substitution instances of $P \vee \neg P$ (which a tautology) are given below:

(i) $(P \to S) \vee \neg(P \to S)$

(ii) $(((P \vee \neg Q) \to R) \rightleftarrows S) \vee \neg(((P \vee \neg Q) \to R) \rightleftarrows S)$.