Theorem 2.14: Lusin's Theorem

Let f be a real valued measurable function on E, then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F and $m(E \sim F) < \varepsilon$.

Proof:

Suppose $m(E) < \infty$.

According to Simple Approximation Theorem, $\exists$ a sequence of simple functions defined on E that converges to f pointwise on E.

Let $n \in \mathbb{N}$,

By proposition 11, replace f by f_n & ε by $\frac{\varepsilon}{2^{n+1}}$.

We may choose a continuous function g_n on $\mathbb{R}$ and a closed set $F_n \subseteq E$ for which $f_n = g_n$ on F_n &

$$m(E \sim F_n) < \frac{\varepsilon}{2^{n+1}} \longrightarrow (1).$$

By Egoroff's theorem, there is a closed set $F_0 \subseteq E$ such that $f_n \longrightarrow f$ uniformly on F_0 &

$$m(E \sim F_0) < \frac{\varepsilon}{2} \longrightarrow (2)$$

Define $F = \bigcap_{n=0}^{\infty} F_n$

Consider $m(E \sim F) = m(E \sim \bigcap_{n=0}^{\infty} F_n) = m(\bigcup_{n=0}^{\infty} E \sim F_n)$

$$= m\left[(E \sim F_0) \cup \bigcup_{n=1}^{\infty} (E \sim F_n)\right]$$

$$\leq \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \varepsilon$$

Now, we note that F is closed.

Each f_n is continuous on F ($\because F \subseteq F_n$ & $f_n = g_n$ on F_n)

Finally. $\{f_n\} \to f$ uniformly on F. ($\because F \subseteq F_0$)

Since the uniform limit of continuous function is continuous. So the restriction of f to F is continuous on F.

Finally there is a continuous function g defined on R whose restriction to F equals to f.

This function g has the required approximation properties.

Hence proved.